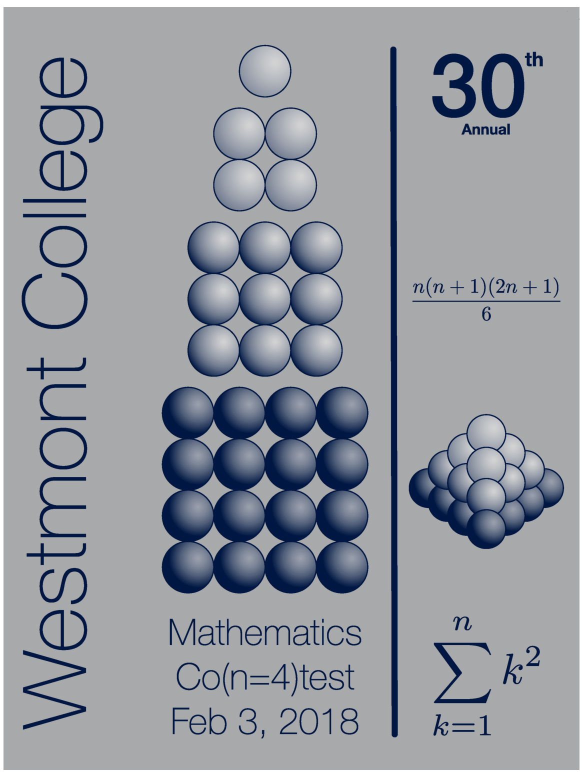
## T-Shirt Explanation: Westmont's 30th Annual Mathematics Contest

The T-shirt for Westmont's 30th annual mathematics contest, shown below and to the left, illustrates that 30 is a *pyramidal number*: it is the sum of the first four squares. Collections of objects corresponding to collections of successive square numbers can be built up to form a pyramid with a square base. The T-shirt depicts 1, 4, 9, and 16 spheres stacked from the bottom up forming such a pyramid. The right-hand pane of the T-shirt gives a formula for the sum of the first $n$ squares:

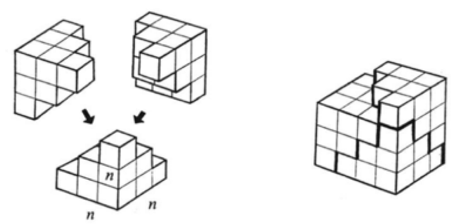
$$1 + 4 + 9 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

When $n = 4$ the sum is $1 + 4 + 9 + 16 = 30$, which checks out: $\frac{4(5)(2\cdot 4+1)}{6} = \frac{4(5)(9)}{6} = \frac{180}{6} = 30$, which is why the bottom portion of the T-shirt has the term Co(n=4)test.

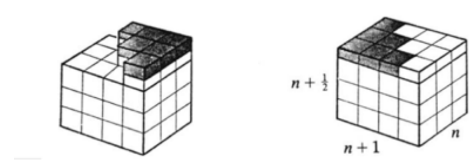
A standard proof of this result is produced with a technique known as mathematical induction. There is, however, a proof using pictures only (no words) that was supplied by Man-Keung Su of the University of Hong Kong. It appeared in the March, 1984 issue of *Mathematics Magazine*.




Begin with three identical representations of the first $n$ squares stacked together. With $n = 3$, each stack has layers of 1, 4, and 9 cubes. Arrange the stacks so they will fit together as shown.




Slice the upper layer through the middle, rotate it, and slide the sliced-off part into the vacant space.




You wind up with a block whose volume is the product of its dimensions: $n(n+1)(n+\frac{1}{2})$.

This volume, however, is three times the volume of any single stack shown in the first stage of this process. The volume of that stack, of course, is the sum of the first $n$ squares, which must equal one-third of the volume of the final block. Thus, the sum of the first $n$ squares must be

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{1}{3}n(n+1)\left(n + \frac{1}{2}\right) = \frac{n(n+1)(2n+1)}{6},$$

which is the expression displayed on the T-shirt.