Westmont College Mathematics Contest Team Exam

Grades 9–10, February 22, 2020

Answer Key

School: ______________________________

Instructions:

Put the name of your school in the space above, and on each solution page.

Calculators are not permitted.

This is a team exam. There are only five questions, so your team will be graded both on the accuracy of your answer and the clarity with which you express your answer. Turn in only one solution per problem per team. Work each problem out on scratch paper at first, then copy the solution you want graded to the space provided below the corresponding question, making sure your writing is legible and coherent. Partial credit will be awarded if you make good progress on a problem, so turn in your work even for problems you did not completely solve. Be sure you justify your answer rather than merely stating what the answer is. Use the back of the paper if you run out of room. When time is up, assemble your answers in order and give them to the exam proctor.

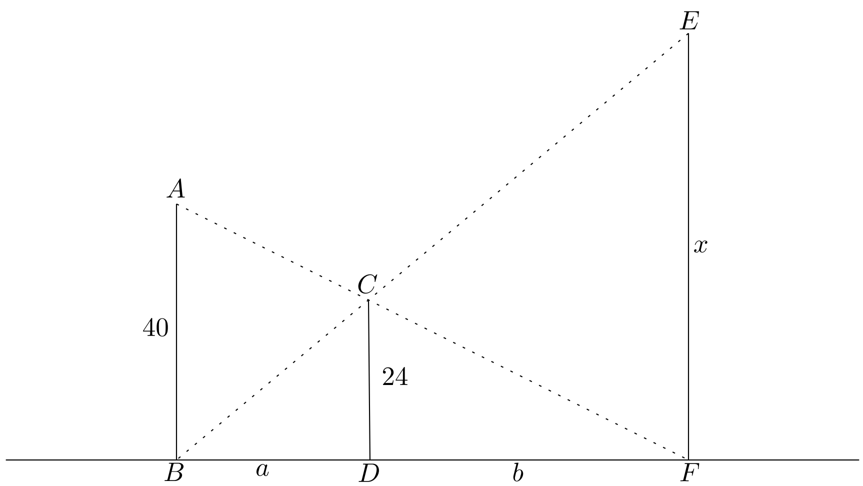
If you finish early you may turn in your answers and watch your classmates from the other grades compete in the college bowl!

Two different-sized vertical poles are in level ground. Laser beams from the top of each pole to the base of the other pole cross at a point 24 feet above ground. If the shorter pole is 40 feet high, how high is the taller pole?

Solution: 60 feet.

Explanation:

In the figure below triangle CBD is congruent to triangle EBF, and triangle CDF is congruent to triangle ABF. Therefore, $\frac{24}{x} = \frac{a}{a+b}$, and $\frac{24}{40} = \frac{b}{a+b}$. Adding gives $\frac{24}{x} + \frac{24}{40} = \frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1$, from which we derive $x = 60$.



Problem 2

Two drivers, A and B, are parked at a distance of 225 miles apart along a straight road. At a certain point in time A starts traveling towards B at a constant speed of x miles per hour. Thirty minutes later B starts traveling towards A at the same constant speed of x miles per hour. Upon meeting, each driver continues to the other's starting point at a constant speed of $x - 10$ miles per hour. If A completes the entire trip in five hours, what was the constant speed x of each driver?

Solution: 50 miles per hour.

Explanation:

Refer to the figure below, and keep in mind that (rate) $\times$ (time) = distance, so time = $\frac{\text{distance}}{\text{rate}}$.

The trip is divided into segments, where A is traveling from left to right, and B is traveling from right to left. For the first 30 minutes ($\frac{1}{2}$ hour) A's speed is x miles per hour, so as the diagram indicates the distance A covers is $\frac{1}{2}x$ miles. The total distance is 225 miles, so at that point (marked with a dot on the diagram) A has $225 - \frac{x}{2}$ miles left to travel.

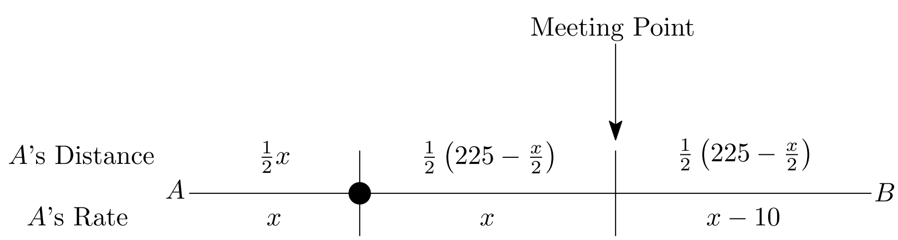
Since B begins traveling at that time, and at the same speed, both A and B will travel half that distance, or $\frac{1}{2}\left(225 - \frac{x}{2}\right)$ miles, when they meet. The speed past that point is $x - 10$ miles per hour. Note that only A's speed (rate) is shown on the diagram.

Because A took 5 hours to complete the entire trip, and the problem implies that $x > 0$, we have

$$5 = \text{time} = \frac{\text{distance}}{\text{rate}} = \frac{\frac{1}{2}x}{x} + \frac{\frac{1}{2}\left(225 - \frac{x}{2}\right)}{x} + \frac{\frac{1}{2}\left(225 - \frac{x}{2}\right)}{x - 10}.$$

Simplifying gives $2x^2 - 109x + 450 = 0$. This expression factors as $(2x - 9)(x - 50) = 0$. We disregard the solution $x = \frac{9}{2}$, as this would give a negative rate of $\frac{9}{2} - 10 = -5.5$ miles per hour after they meet, which is impossible if both A and B continue towards the other's starting point.

Therefore, the constant speed is $x = 50$ miles per hour.



Problem 3

Let a, b, c, and d be integers with $a < 2b$, $b < 3c$, and $c < 4d$. If $d < 100$, what is the largest possible value for a?

Solution: The largest possible value is $a = 2367$.

Explanation:

Obviously, to maximize a, d must be as large as possible. The largest possible value for d is 99, so $c < (4)(99) = 396$. That is, the largest possible value for c is 395. Likewise, $b < (3)(395) = 1185$, so the largest possible value for d is 1184. Finally, because $a < (2)(1184) = 2368$, the largest possible value for a is 2367.

Given the sequence of numbers 2, x, y, 128, z, ...

1. What do x, y, and z equal if the sequence is arithmetic?

2. What do x, y, and z equal if the sequence is geometric?

An *arithmetic* sequence has the property that the difference of its successive terms is constant, and a *geometric* sequence has the property that the quotient of its successive terms is constant.

Solutions:

1. $\underline{\quad x = 44,\ y = 86,\ z = 170 \quad}$.

2. $\underline{\quad x = 8,\ y = 32,\ z = 512 \quad}$.

Explanation:

1. For the arithmetic sequence, $x - 2 = y - x = 128 - y$. Thus, we get two equations with two unknowns: $x - 2 = y - x$, and $y - x = 128 - y$. These equations can be rewritten as follows:

$$\begin{aligned} 2x - y &= 2 \\ -x + 2y &= 128. \end{aligned}$$

Solving gives $x = 44$ and $y = 86$, so that the (constant) difference of consecutive terms equals $y - x = 86 - 44 = 42$. Therefore, $z = 128 + 42 = 170$.

2. For the geometric sequence, $\frac{x}{2} = \frac{y}{x} = \frac{128}{y}$. Again, we get two equations with two unknowns: $x^2 = 2y$, and $y^2 = 128x$. The first equation gives $y = \frac{x^2}{2}$. Substituting this value of y into the second equation produces $(\frac{x}{2})^2 = 128x$, or $\frac{x^4}{4} = 128x$. As $x \neq 0$ this last equation simplifies to $x^3 = 512$, so $x = 8$. Then, because $y = \frac{x^2}{2}$, we get $y = 32$, so that the (constant) quotient of consecutive terms equals $\frac{y}{x} = \frac{32}{8} = 4$. Therefore, $z = 128 \times 4 = 512$.

Kay is standing at location $(0,0)$ and moves in stages to location $(6,6)$. At each stage she can, with equal probability, either increase her x-coordinate by 1, or increase her y coordinate by 1, but not both. Expressed as a fraction in lowest terms, what is the probability that she will pass through location $(2,3)$?

Solution: The probability is $\frac{175}{462}$.

Explanation:

There are $\frac{12!}{(6!)(6!)} = 924$ paths from $(0,0)$ to $(6,6)$, as each path corresponds to an ordered arrangement of the twelve symbols $EEEEEENNNNNN$, where "E" represents heading East (increasing the x-coordinate by 1), and "N" represents heading North (increasing the y-coordinate by 1).

Using similar reasoning, we see that there are $\frac{5!}{(2!)(3!)} = 10$ paths from $(0,0)$ to $(2,3)$, and $\frac{7!}{(4!)(3!)} = 35$ paths from $(2,3)$ to $(6,6)$.

Thus, there are $10 \times 35 = 350$ paths from $(0,0)$ to $(6,6)$ that pass through $(2,3)$, and the probability of taking one of them is $\frac{350}{924} = \frac{175}{462}$.