Westmont College Mathematics Contest Team Exam

Grades 11–12, February 22, 2020

Answer Key

School: ______________________________

Instructions:

Put the name of your school in the space above, and on each solution page.

Calculators are not permitted.

This is a team exam. There are only five questions, so your team will be graded both on the accuracy of your answer and the clarity with which you express your answer. Turn in only one solution per problem per team. Work each problem out on scratch paper at first, then copy the solution you want graded to the space provided below the corresponding question, making sure your writing is legible and coherent. Be sure you justify your answer rather than merely stating what the answer is. Use the back of the paper if you run out of room. When time is up, assemble your answers in order and give them to the exam proctor.

If you finish early you may turn in your answers and watch your classmates from the other grades compete in the college bowl!

The number $n!$ (read as "n factorial") is defined by

$$n! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times n.$$

The number $5! = 120$ has only one trailing zero; the number $15! = 1{,}307{,}674{,}368{,}000$ has three trailing zeros. In honor of the current year, find how many trailing zeros the number 2020! has.

Solution: 503 trailing zeros.

Explanation:

The number of trailing zeros a number has is determined by how many factors of 10 comprise the number. A factor of 10 will occur whenever a "2" can be paired with a "5" in the number's prime factorization. For the number $2020! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 2020$, every other term contains a factor of two. Thus, the question reduces to how many factors of 5 there are in 2020! Clearly, these factors arise from the terms $5, 10, 15, 20, 25, 30$, and so forth, that make up the factorial of 2020. Of course, terms that are multiples $5^2 = 25$ each contribute two factors of 5, terms that are multiples of $5^3 = 125$ each contribute three factors of 5, and terms that are multiples of $5^4 = 625$ each contribute four factors of 5. Since $5^5 = 3125$, and $3125 > 2020$, no higher powers of 5 (beyond $5^4 = 625$) occur in terms making up the factorial of 2020.

The total number of factors of 5 can thus be determined by four simple calculations:

1. $\frac{2020}{5} = 404$, so there are 404 terms in the factorial of 2020 that have a factor of 5.

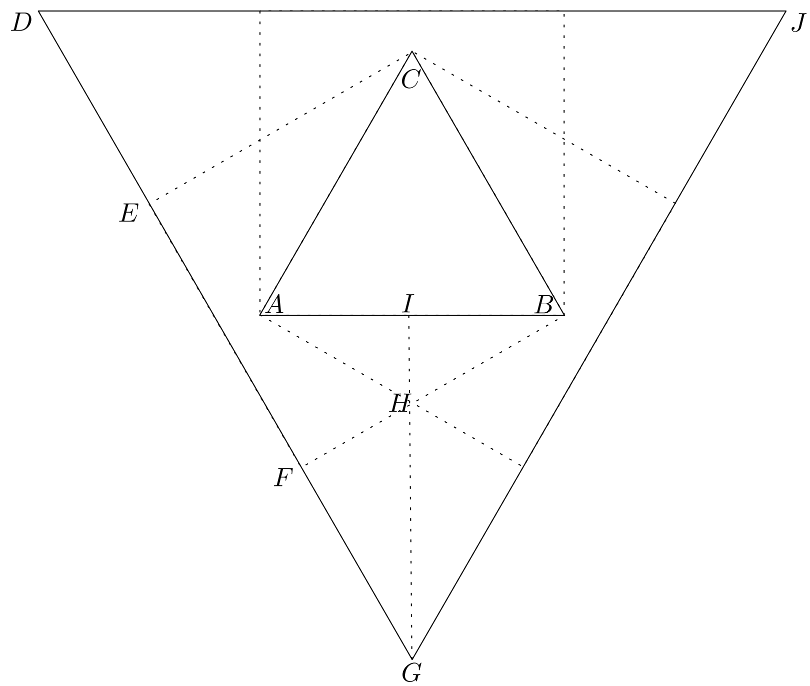
2. $\frac{2020}{25} = 80.8$, so among the 404 terms in the factorial of 2020 that have a factor of 5, there are 80 terms that also have a factor of $5^2 = 25$. In other words, there are 80 terms that each contribute an additional factor of 5.

3. $\frac{2020}{125} = 16.16$, so among the 80 terms in the factorial of 2020 that have a factor of 25, there are 16 terms that also have a factor of $5^3 = 125$. These 16 terms each contribute an additional factor of 5 as well.

4. $\frac{2020}{625} = 3.232$, so among the 16 terms in the factorial of 2020 that have a factor of 125, there are 3 terms that also have a factor of $5^4 = 625$. These 3 terms collectively contribute the final 3 factors of 5.

Therefore, the total number of trailing zeros in 2020! is $404 + 80 + 16 + 3 = 503$.

In the diagram below equilateral triangle ABC has side length 2. On each of its sides a square is drawn containing the triangle and having that side as one of its edges. What is the side length of the smallest triangle containing these three squares?



Solution: $4\sqrt{3} - 2$.

Explanation:
The minimal bounding triangle is shown as triangle DGJ. Because ABC is equilateral, $\angle CBA$ is 60 degrees. Also, $\angle CBF$, being an angle of a square, is a right angle. Triangle HIB is a 30-60-90 degree right triangle, with leg IB equal to 1. Thus, $HB = \frac{2}{\sqrt{3}}$, so $FH = 2 - \frac{2}{\sqrt{3}}$. Similarly, $\angle FHG = \angle IHB = 60$ degrees, so triangle GFH is a 30-60-90 degree right triange with leg $FH = 2 - \frac{2}{\sqrt{3}}$. Thus, $FG = \sqrt{3} \cdot FH = 2\sqrt{3} - 2$. A symmetric argument shows that $DE = 2\sqrt{3} - 2$, so we finally conclude that $DG = 2(2\sqrt{3} - 2) + 2 = 4\sqrt{3} - 2$.

Let $S = \{1, 4, 9, 16, 25, \ldots\}$ be the set of squares of positive integers, and let p be the 22nd prime number (in honor of today's date, February 22). Find the square t in S so that $t + p$ is also in S.

Solution: $t = 1521$

Explanation:

Set $t = x^2$. Because 79 is the 22nd prime number, the problem reduces to solving the equation $x^2 + 79 = y^2$. Rewriting gives $y^2 - x^2 = 79$, or $(y + x)(y - x) = 79$. Since x and y are positive integers and 79 is a prime number, the last equation implies that $y + x = 79$ and $y - x = 1$. Solving for x gives $x = 39$, so $t = x^2 = 39^2 = 1521$ (and $y^2 = 1521 + 79 = 1600 = 40^2$).

What is the value of the product $\left(\sin\frac{\pi}{32}\right)\left(\cos\frac{\pi}{32}\right)\left(\cos\frac{\pi}{16}\right)\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right)$?

Solution: $\frac{1}{16}$

Explanation:

Make use of the well-known trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$. Then

$$\begin{aligned}
\left(\sin\frac{\pi}{32}\right)\left(\cos\frac{\pi}{32}\right)\left(\cos\frac{\pi}{16}\right)\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right) &= \frac{1}{2}\left[2\left(\sin\frac{\pi}{32}\right)\left(\cos\frac{\pi}{32}\right)\right]\left(\cos\frac{\pi}{16}\right)\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right)\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left[2\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right)\right]\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[2\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)\right]\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[2\left(\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)\right]\\
&= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{2}\right)\\
&= \frac{1}{16}.
\end{aligned}$$

Using standard integration techniques it can be shown that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$

1. Assuming the above statement is correct, explain why it follows that, contrary to the belief of many, $\pi \neq \frac{22}{7}$. In fact, explain why the above statement implies that $\pi < \frac{22}{7}$.

2. Use appropriate integration techniques to show that, indeed, $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$

Explanations:

1. All the terms in the integrand are strictly positive except when $x = 0$ and $x = 1$. The integral, then, is measuring the area under the graph of a positive function for $0 < x < 1$. Thus, the integral must be positive, so that $0 < \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$. Therefore, $\pi < \dfrac{22}{7}$.

2. In the numerator expand $(1-x)^4$ and multiply the result by x^4 to get $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$. Then divide this expression by $1 + x^2$ to get $x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$. Finally, evaluate the resulting integral:

$$\begin{aligned}
\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right) dx \\
&= \left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\arctan x\right)\Bigg|_0^1 \\
&= \left(\frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\arctan 1\right) - 0 \\
&= \frac{3 - 14 + 21 - 28 + 84}{21} - 4\left(\frac{\pi}{4}\right) \\
&= \frac{66}{21} - \pi \\
&= \frac{22}{7} - \pi.
\end{aligned}$$